SYMMETRY AND UNIQUENESS OF SOLUTIONS TO SOME LIOUVILLE-TYPE PROBLEMS: ASYMMETRIC SINH-GORDON EQUATION, COSMIC STRING EQUATION AND TODA SYSTEM

CHANGFENG GUI, ALEKS JEVNIKAR, AMIR MORADIFAM

Abstract. We are concerned with symmetry and uniqueness results for three classes of Liouville-type problems defined on bounded domains arising in geometry and mathematical physics: asymmetric Sinh-Gordon equation, cosmic string equation and Toda system. In the spirit of the Sphere covering inequality we provide a series of results under suitable assumptions on the mass associated to these problems.

1. Introduction

We start by considering the following version of the asymmetric Sinh-Gordon equation

\begin{align}
-\Delta u &= \rho \frac{e^u + \frac{\alpha}{|\alpha|} e^{\alpha u}}{\int_{\Omega} (e^u + e^{\alpha u}) \, dx} \quad \text{in } \Omega, \\
\end{align}

\begin{align}
\int_{\Omega} (e^u + e^{\alpha u}) \, dx \quad \text{in } \Omega,
\end{align}

where $\alpha \in [-1, 1], \alpha \neq 0$, $\rho > 0$ is a parameter and $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$. Equation (1) is known also as Neri’s mean field equation and arises in the context of the statistical mechanics description of 2D-turbulence introduced in [35]. In the model where the circulation number density is subject to a probability measure, under a stochastic assumption on the vortex intensities one obtains the following equation, see [34]:

\begin{align}
-\Delta u &= \rho \int_{[-1, 1]} \beta e^{\beta u} \mathcal{P}(d\beta) \quad \text{in } \Omega, \\
\end{align}

\begin{align}
\int_{[-1, 1] \times \Omega} e^{\beta u} \mathcal{P}(d\beta) \quad \text{on } \partial \Omega,
\end{align}

where $u$ stands for the stream function of a turbulent Euler flow, $\mathcal{P}$ is a Borel probability measure defined in $[-1, 1]$ describing the point vortex intensity distribution and $\rho > 0$ is a physical constant associated to the inverse temperature. Equation (1) is related to the latter model when $\mathcal{P}$ is supported in two points.

2000 Mathematics Subject Classification. 35J61, 35R01, 35A02, 35B06.

Key words and phrases. Geometric PDEs, Sinh-Gordon equation, Cosmic string equation, Toda system, Sphere covering inequality, Symmetry results, Uniqueness results.

The first author is partially supported by a Simons Foundation Collaborative Grant (Award #199305) and NSFC grant No 11371128. The second author is supported by PRIN12 project: Variational and Perturbative Aspects of Nonlinear Differential Problems and FIRB project: Analysis and Beyond.
On the other hand, a deterministic assumption on the vortex intensities yields the following model, see [43]:

\[
\begin{align*}
\begin{cases}
-\Delta u &= \rho \left( \frac{e^u}{\int_\Omega e^u \, dx} + \frac{\alpha}{|\alpha|} \frac{e^{\alpha u}}{\int_\Omega e^{\alpha u} \, dx} \right) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\end{align*}
\]

Concerning the analysis of the latter equation we refer the interested readers to [1, 18, 19, 20, 21, 23, 24, 25, 26, 39, 41]. The arguments presented here do not apply to the problem (3) and we postpone its analysis to the forthcoming papers.

Observe that by taking \( \alpha = -1 \) in (1) we end up with the standard Sinh-Gordon equation, while for \( P \) supported in a single point we derive the standard mean field equation

\[
\begin{align*}
\begin{cases}
-\Delta u &= \rho e^u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

which is related to the prescribed Gaussian curvature problem and Euler flows, see [3, 45] and [10, 28], respectively. The latter equation has been widely studied and we refer to the surveys [32, 47]. We point out that recently the authors in [16, 17] provide symmetry and uniqueness results for the latter equation by using the Sphere covering inequality obtained in [16]. The topic of our paper will be in this spirit.

Returning to (1) some partial existence results and blow-up analysis was carried out in [40, 42], while a complete existence result for (2) with \( \text{supp } P \subset [0, 1] \) was given in [14]. On the other hand, we are not aware of symmetry or uniqueness results for the latter equation with the only exception of [44] where (3) is considered. We present here some results in this direction by suitable natural assumptions both on the parameter \( \rho \) and the domain \( \Omega \). Due to the different features of the problem (2) depending on whether \( \text{supp } P \subset [0, 1] \) or \( \text{supp } P \subset [-1, 1] \), we will distinguish between the latter two cases. In the first situation we may rewrite (1) as

\[
\begin{align*}
\begin{cases}
-\Delta u &= \rho \frac{e^u}{\int_\Omega (e^u + e^{\alpha u}) \, dx} \quad \text{in } \Omega, \\
u &= g(x) \geq 0 \quad \text{on } \partial \Omega,
\end{cases}
\end{align*}
\]

with \( \alpha \in (0, 1) \) and \( g \equiv 0 \). The first result is the following.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a simply-connected domain and \( g \in C(\partial \Omega) \) be a non-negative function. Suppose \( \rho \leq 4\pi \). If \( u_1 \) and \( u_2 \) are two solutions of (5) such that

\[
\int_\Omega (e^{u_1} + e^{\alpha u_1}) \, dx = \int_\Omega (e^{u_2} + e^{\alpha u_2}) \, dx,
\]

then \( u_1 \equiv u_2 \).

**Corollary 1.2.** Let \( 0 \in \Omega \subset \mathbb{R}^2 \) be a simply-connected domain, \( g \in C(\partial \Omega) \) in non-negative, and \( \rho \leq 4\pi \). Suppose that \( \Omega \) and \( g \) are evenly symmetric about a line. Then, \( u \) is evenly symmetric about that line. In particular, if \( \Omega \) is radially symmetric and \( g \) is a non-negative constant, then \( u \) is radially symmetric.
We will exploit the fact that for $\text{supp} \mathcal{P} \subset [0, 1]$ equation (2) shares some features with the mean field equation (4): by suitably rewriting of the problem we will apply the Sphere covering inequality of [16] to get the desired properties.

**Remark 1.3.** The argument of Theorem 1.1 can be suitably adapted to treat the more general case where the probability measure $\mathcal{P}$ in the model problem (2) with $\text{supp} \mathcal{P} \subset [0, 1]$ supported in $(m + 1)$ points, i.e.

$$\begin{cases}
-\Delta u = \rho \frac{e^u + e^{a_1 u} + \cdots + e^{a_m u}}{\int_{\Omega} (e^u + e^{a_1 u} + \cdots e^{a_m u}) \, dx} \quad \text{in } \Omega, \\
u = g \geq 0 \quad \text{on } \partial \Omega,
\end{cases}$$

with $a_i \in (0, 1)$ for all $i$. In this case the same conclusions of Theorem 1.1 hold true by assuming $\rho \leq \frac{8\pi}{m + 1}$ and

$$\int_{\Omega} (e^{u_1} + e^{a_1 u_1} + \cdots e^{a_m u_1}) \, dx = \int_{\Omega} (e^{u_2} + e^{a_1 u_2} + \cdots e^{a_m u_2}) \, dx.$$  

In particular, Corollary 1.2 yields symmetry properties to solutions of the above equation. The case where $a_i > 1$ for some $i$ can be carried out as well and we refer to Remark 1.5 for more details.

On the other hand, for general $\text{supp} \mathcal{P} \subset [-1, 1]$ case the problem (2) substantially differs from the standard equation (4). In this case we may rewrite (1) as

$$\begin{cases}
-\Delta u = \rho \frac{e^u - e^{-au}}{\int_{\Omega} (e^u + e^{-au}) \, dx} \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial \Omega,
\end{cases}$$

with $a \in (0, 1]$. Observe that $u \equiv 0$ is a solution of the latter problem. We indeed show that for some ranges of $\rho$ the trivial solution is the only solution.

**Theorem 1.4.** Suppose $\rho \leq \frac{8\pi}{1 + a}$ and $\Omega$ simply-connected. Then, equation (7) admits only the trivial solution $u \equiv 0$.

The idea is in the same spirit of the argument of the Sphere Covering Inequality [16], see Section 2: roughly speaking, letting $v_1 = u$, $v_2 = -au$ we will consider a symmetrization of $v_2 - v_1$ with respect to two suitable measures to get the conclusion.

**Remark 1.5.** Let us point out that in the equations (5), (7) we are considering $a < 1$ (resp. $a \leq 1$) due to the physical motivations. However, we can treat the case $a > 1$ as well: letting $v = au$ in (5) we may rewrite the latter equation in a form to which we can apply Theorem 1.2 with a new parameter $\tilde{\rho} = a \rho$. Therefore, the conclusions of Theorem 1.1 and Corollary 1.2 still hold true for $\rho \leq \frac{4\pi}{a}$, $a > 1$. On the other hand, one can easily see from the proof of Theorem 1.4 that the assumption $a \leq 1$ is not needed: hence, we get the same conclusion for $a > 1$.

**Remark 1.6.** The same arguments clearly apply to the following version of (1):

$$\begin{cases}
-\Delta u = e^u + \frac{a}{|a|}e^{au} \quad \text{in } \Omega, \\
u = g(x) \geq 0 \quad \text{on } \partial \Omega.
\end{cases}$$
Following the proofs of Theorems 1.2, 1.4 one can check the assumptions $\rho \leq 4\pi$, $a < 1$ (resp. $\rho \leq 4\pi/a$, $a > 1$) in Theorem 1.2 and $\rho \leq 8\pi/(1+a)$ in Theorem 1.4 are replaced by $$\int_{\Omega} e^u \, dx \leq 4\pi, \quad a < 1 \quad \text{ (resp. } \int_{\Omega} e^{au} \, dx \leq \frac{4\pi}{a}, \quad a > 1 \text{)}$$ and $$\int_{\Omega} (e^u + e^{-au}) \, dx \leq \frac{8\pi}{1+a},$$ respectively.

Finally, let us give the following remark concerning the sharpness of the above results.

**Remark 1.7.** Consider for simplicity the standard Sinh-Gordon equation with $\alpha = -1$ in (1). Even though its associated energy functional is coercive for $\rho < 8\pi$, see [42], it is not clear whether we can extend Theorem 1.1, Corollary 1.2 and Theorem 1.4 up to $\rho \leq 8\pi$ (as it holds for the standard mean field equation (4)). Indeed, the result of [44] in which the authors exhibit non-trivial solutions for (3) with $\rho < 8\pi$ (see at the end of Section 2 in the latter paper), suggests this is not the case.

We will next pass to the following problem to which we will refer as the cosmic string equation:

$$
\begin{cases}
-\Delta u = e^{au} + h(x)e^u & \text{in } \Omega, \\
u = g(x) \geq 0 & \text{on } \partial\Omega,
\end{cases}
$$

with $a > 0$, $0 \in \Omega \subset \mathbb{R}^2$ and $h$ is of the form

$$h(x) = e^{-4\pi NG_0(x)},$$

where $N \in \mathbb{N}$ and $G_0$ is the Green’s function with pole at 0, i.e.

$$
\begin{cases}
-\Delta G_0(x) = \delta_0 & \text{in } \Omega, \\
G_0(x) = 0 & \text{on } \partial\Omega.
\end{cases}
$$

Observe that

\[ h > 0 \text{ in } \Omega \setminus \{0\} \] and \[ h(x) \equiv |x|^{2N} \text{ near } 0. \]

Equation (8) describes the behavior of selfgravitating cosmic strings for a massive W-boson model coupled with Einstein’s equation where $a$ is a physical parameter and $N$ the string’s multiplicity, see [2, 36, 50]. Observe that for $a = 1$ (8) is related to the Gaussian curvature with conic singularities, see [47].

Many results concerning (8) have been established especially for the full plane case: we refer to [12, 13, 50] for existence results, to [36, 37] for what concerns symmetry issues and to [48] for blow-up analysis. In particular, in [36, 37] the authors provide necessary and sufficient for the solvability of (8) in the full plane in the context of radially symmetric solutions, depending on the values of the total mass $\beta = \int_{\mathbb{R}^2} (e^{au} + |x|^{2N}e^u) \, dx$. If we allow $N \in (-1, 0]$ all the solutions to (8) are radially symmetric under suitable assumptions on the domain $\Omega$ by a moving
plane argument. However, it remains an open problem if some of the results in [36, 37] are sharp for the non-radial framework. We prove the following result.

**Theorem 1.8.** Let $\Omega \subset \mathbb{R}^2$ be a simply-connected domain, $a > 0$, $N \geq 0$ and $g \in C(\partial \Omega)$ be non-negative. Suppose $u_1$ and $u_2$ are two distinct solutions of (8) such that either

$$(11) \quad \int_{\Omega} (e^{a u_1} + e^{a u_2}) \, dx \leq \frac{8 \pi}{a} \quad \text{if} \quad a > 1,$$

or

$$(12) \quad \int_{\Omega} (e^{u_1} + e^{u_2}) \, dx \leq 8 \pi \quad \text{if} \quad a < 1.$$

Then $u_1$ and $u_2$ can not intersect, i.e. either

$$(13) \quad u_2 > u_1 \quad \text{or} \quad u_2 < u_1 \quad \text{in} \quad \Omega.$$

**Corollary 1.9.** Let $0 \in \Omega \subset \mathbb{R}^2$ be a simply-connected domain, $a > 0$, $N \geq 0$ and $g \in C(\partial \Omega)$ be non-negative. Suppose $\Omega$ and $g$ are evenly symmetric about a line passing through the origin, and either

$$(14) \quad \int_{\Omega} e^{a u} \, dx \leq \frac{4 \pi}{a} \quad \text{if} \quad a > 1,$$

or

$$(15) \quad \int_{\Omega} e^{a u} \, dx \leq 4 \pi \quad \text{if} \quad a < 1.$$

Then, $u$ is evenly symmetric about that line. In particular, if $\Omega$ is radially symmetric about the origin and $g$ is a non-negative constant, then $u$ is radially symmetric about the origin.

The proof is roughly based on a simple manipulation of equation (8) and on the Sphere Covering Inequality [16].

**Remark 1.10.** Similar arguments as in Theorem 1.8 and Corollary 1.9 yielding the same conclusions can be carried out for the following general equation (we refer to [37] for the motivations):

$$
\begin{cases}
-\Delta u = \sum_{i=0}^{m} h_i(x) e^{a_i u} & \text{in} \ \Omega, \\
\quad u = g(x) \geq 0 & \text{on} \ \partial \Omega,
\end{cases}
$$

where $a_i > 0$ and

$$h_i(x) = e^{-4 \pi N_i G_0(x)},$$

with $N_i \geq 0$ for all $i$. Let $a_M = \max_i \{a_i\}$. One can check the assumptions (11) and (13) (where $m = 1$) are replaced respectively by

$$(16) \quad \int_{\Omega} (e^{a_M u_1} + e^{a_M u_2}) \, dx \leq \frac{16 \pi}{a_M (m + 1)}.$$

and

$$(17) \quad \int_{\Omega} e^{a_M u} \, dx \leq \frac{8 \pi}{a_M (m + 1)}.$$
Finally, we will be concerned with the following class of Liouville-type systems:

\[
\begin{cases}
-\Delta u_1 = Ae^{u_1} - Be^{u_2} & \text{in } \Omega, \\
-\Delta u_2 = B'e^{u_2} - A'e^{u_1} & \text{in } \Omega, \\
u_1 = u_2 = g(x) & \text{on } \partial \Omega,
\end{cases}
\]

with \( g \in C(\partial \Omega) \) and such that

\[
A, A', B, B' \geq 0, \quad A + A' = B + B' := M > 0,
\]

hold true. Observe that we allow some of the above coefficients to be zero.

The latter system is deeply connected both with geometry and mathematical physics. For example, by taking \( A = B' = 2, \ B = A' = 1 \) we recover the \( 2 \times 2 \) Toda system which has been extensively studied in the literature: one one side it appears in the description of holomorphic curves in \( \mathbb{C}P^N \), see [9, 11, 31], on the other hand it arises in the non-abelian Chern-Simons theory in the context of high critical temperature superconductivity, see [15, 49, 50]. We point out that the case \( A = B' = 1, \ B = A' = \tau \) and with a singular source was considered in [38] on unbounded domains.

For what concerns Toda-type systems we refer to [27, 29, 30] for blow-up analysis, to [31] for classification issues and to [7, 22, 33] for existence results. On the other hand, we are not aware of symmetry or uniqueness results for Liouville-type systems alike (14). In this direction we provide the following result.

**Theorem 1.11.** Let \((u_1, u_2)\) be a solution of (14) and (15). Let \( M \) be as defined in (15). Suppose

\[\int_\Omega (e^{u_1} + e^{u_2}) \, dx \leq \frac{8\pi}{M}\]

and \( \Omega \) simply-connected. Then \( u_1 \equiv u_2 \equiv u \), where \( u \) is the unique solution to

\[
\begin{cases}
-\Delta u = De^u & \text{in } \Omega, \\
u = g(x) & \text{on } \partial \Omega,
\end{cases}
\]

and \( D := A - B = B' - A' \).

**Remark 1.12.** For Toda-type systems where \( A = B' = 2, \ B = A' = 1 \), the above result asserts that if

\[\int_\Omega (e^{u_1} + e^{u_2}) \, dx \leq \frac{8\pi}{3}\]

and \( \Omega \) simply-connected, then \( u_1 \equiv u_2 \equiv u \), where \( u \) is the unique solution to

\[
\begin{cases}
-\Delta u = e^u & \text{in } \Omega, \\
u = g(x) & \text{on } \partial \Omega.
\end{cases}
\]

Arguing as in the proof of the Sphere Covering Inequality [16], see Section 2, we will consider a symmetrization of \( u_2 - u_1 \) with respect to two suitable measures to get the latter result. The uniqueness property will then follow by applying the Sphere Covering inequality to the scalar equation.
A similar argument can be carried out for the following singular version of (14):

\[
\begin{cases}
-\Delta u_1 = Ae^{u_1} - Be^{u_2} - 4\pi \alpha \delta_0 \\
-\Delta u_2 = B'e^{u_2} - A'e^{u_1} - 4\pi \alpha \delta_0 \\
u_1 = u_2 = g(x)
\end{cases}
\] in \(\Omega\),

\[
\begin{cases}
u_1 = u_2 = g(x)
\end{cases}
\] on \(\partial \Omega\),

where \(\alpha \geq 0\) and \(0 \in \Omega\). Recall the definitions of \(M, D\) in (15) and in Theorem 1.11, respectively. By using the Green’s function \(G_0\) with pole in 0 as in (10) we may consider

\[
u_i(x) = u(x) + 4\pi \alpha G_0(x).
\]

which satisfy

\[
\begin{cases}
-\Delta \tilde{u}_1 = Ah(x)e^{\tilde{u}_1} - Bh(x)e^{\tilde{u}_2} \\
-\Delta \tilde{u}_2 = B'h(x)e^{\tilde{u}_2} - A'h(x)e^{\tilde{u}_1} \\
\tilde{u}_1 = \tilde{u}_2 = g(x)
\end{cases}
\] in \(\Omega\),

\[
\begin{cases}
\tilde{u}_1 = \tilde{u}_2 = g(x)
\end{cases}
\] on \(\partial \Omega\),

with \(h(x) = e^{-4\pi \alpha G_0(x)}\). We have the following result.

**Theorem 1.13.** Let \((u_1, u_2)\) be a solution of (16) with \(\alpha \geq 0\) and (15). Let \(\tilde{u}_i\) be as in (17). Suppose

\[
\int_{\Omega} \left(e^{\tilde{u}_1} + e^{\tilde{u}_2}\right) \, dx \leq \frac{8\pi}{M}
\]

and \(\Omega\) simply-connected. Then \(u_1 \equiv u_2 \equiv u\), where \(u\) is the unique solution to

\[
\begin{cases}
-\Delta u = Du^u - 4\pi \alpha \delta_0 \\
u = g(x)
\end{cases}
\] in \(\Omega\),

\[
\begin{cases}
u = g(x)
\end{cases}
\] on \(\partial \Omega\).

We remark now a possible generalization of the results we obtained so far.

**Remark 1.14.** We point out that all the results we got hold for multiply-connected domains as well provided we consider constant boundary conditions. This follows by suitably adapting the proofs and by using the Sphere Covering Inequality in Theorem 2.5 for multiply-connected domains, see Remark 2.6.

The plan of the paper is the following: in Section 2 we recall the main ingredients of the Sphere Covering Inequality, in Section 3 we present the strategy to prove both the uniqueness result of Theorem 1.1, the symmetry result of Corollary 1.2 and the uniqueness result of Theorem 1.4, in Section 4 we show how to get the no intersection property of Theorem 1.8 and the symmetry property of Corollary 1.9 and in Section 5 we provide the proof of the uniqueness result of Theorems 1.11, 1.13.

**Notation**

The symbol \(B_r(p)\) will denote the open metric ball of radius \(r\) and center \(p\). Where there is no ambiguity, with a little abuse of notation we will write \(x\) and \(dx\) to denote \((x, y) \in \mathbb{R}^2\) and the integration with respect to \((x, y)\), respectively.
2. The Sphere Covering Inequality

In this section we recall the main ingredients of the Sphere Covering Inequality proved in [16] since we will need them in the sequel. Roughly speaking, the latter result asserts that the total area of two distinct surfaces with Gaussian curvature equal to 1, conformal to the Euclidean unit disk with the same conformal factor on the boundary, must cover the whole unit sphere after a proper rearrangement. We refer to [16] for full details. Let us start by recalling the standard Bol’s isoperimetric inequality [8].

Proposition 2.1. Let $\Omega \subset \mathbb{R}^2$ be a simply-connected set and $u \in C^2(\Omega)$ be such that
\[
\Delta u + e^u \geq 0 \quad \text{and} \quad \int_{\Omega} e^u \, dx \leq 8\pi.
\]
Then, for any $\omega \subset \subset \Omega$ of class $C^1$ it holds
\[
\left( \int_{\partial \omega} e^{\frac{2}{n}} \, d\sigma \right)^2 \geq \frac{1}{2} \left( \int_{\omega} e^u \, dx \right) \left( 8\pi - \int_{\omega} e^u \, dx \right).
\]

The basic function satisfying the above properties and which will be used in the sequel is the following:
\[
U_\lambda(x) = -2 \ln \left( 1 + \frac{\lambda^2 |x|^2}{8} \right) + 2 \ln \lambda,
\]
for $\lambda > 0$. Observe that
\[
\Delta U_\lambda + e^{U_\lambda} = 0 \quad \text{and} \quad \int_{B_r(0)} e^{U_\lambda} \, dx = 8\pi \frac{\lambda^2 r^2}{8 + \lambda^2 r^2},
\]
for all $r > 0$.

The idea is now to consider symmetric rearrangements with respect to two measures: more precisely, let $w \in C^2(\Omega)$ be such that
\[
\Delta w + e^w \geq 0.
\]
Then, any function $\phi \in C^2(\Omega)$ can be equimeasurably rearranged with respect to the measures $e^w \, dx$ and $e^{U_\lambda} \, dx$, see [4]. Indeed, for $t > \min_{x \in \Omega} \phi(x)$ let $B_t^*$ be the ball centered at the origin such that
\[
\int_{B_t^*} e^{U_\lambda} \, dx = \int_{\{\phi > t\}} e^w \, dx.
\]
Then, if we let $\phi^* : B_t^* \to \mathbb{R}$ to be $\phi^*(x) = \sup\{t \in \mathbb{R} : x \in B_t^*\}$, it holds that $\phi^*$ is a symmetric equimeasurable rearrangement of $\phi$ with respect to the measures $e^w \, dx$ and $e^{U_\lambda} \, dx$:
\[
\int_{\{\phi^* > t\}} e^{U_\lambda} \, dx = \int_{\{\phi > t\}} e^w \, dx,
\]
for all $t > \min_{x \in \Omega} \phi(x)$. Moreover, by using the Bol’s inequality stated in Proposition 2.1 we get the following estimate on the gradient of the rearrangement, see [16].
Proposition 2.2. Let \( w \in C^2(\Omega) \) be such that it satisfies (19) with \( \Omega \subset \mathbb{R}^2 \) simply-connected. Let \( U_\lambda \) be as in (18). Suppose \( \phi \in C^2(\Omega) \) is such that \( \phi \equiv C \) on \( \partial \Omega \). Then, letting \( \phi^* \) be the equimeasurable symmetric rearrangement of \( \phi \) with respect to the measures \( e^w \, dx \) and \( e^{U_\lambda} \, dx \), see (20), it holds for all \( t > \min_{x \in \Omega} \phi(x) \)

\[
\int_{\{\phi^*=t\}} |\nabla \phi^*| \, d\sigma \leq \int_{\{\phi=t\}} |\nabla \phi| \, d\sigma.
\]

To proceed further we need the following counterpart of the Bol’s inequality in the radial setting, see [16, 46].

Proposition 2.3. Let \( \psi \in C^{0,1}(\overline{B_R(0)}) \) be a strictly decreasing radial function satisfying

\[
\int_{\partial B_r(0)} |\nabla \psi| \, d\sigma \leq \int_{B_r(0)} e^\psi \, dx \quad \text{for a.e. } r \in (0, R) \quad \text{and} \quad \int_{B_R(0)} e^\psi \, dx \leq 8\pi.
\]

Then, it holds

\[
\left( \int_{\partial B_R(0)} e^{\frac{\psi}{2}} \, d\sigma \right)^2 \geq \frac{1}{2} \left( \int_{B_R(0)} e^\psi \, dx \right) \left( 8\pi - \int_{B_R(0)} e^\psi \, dx \right).
\]

The crucial ingredient then relates strictly decreasing radial functions \( \psi \) with two standard bubbles \( U_{\lambda_1}, U_{\lambda_2} \) defined in (18) with \( \lambda_2 > \lambda_1 \), such that \( \psi = U_{\lambda_1} = U_{\lambda_2} \) on \( \partial B_R(0) \). The first part relies on the latter Proposition 2.3, while the last property follows by direct computations, see [16].

Proposition 2.4. \( U_{\lambda_1}, U_{\lambda_2} \) defined in (18) with \( \lambda_2 > \lambda_1 \). Let \( \psi \in C^{0,1}(\overline{B_R(0)}) \) be a strictly decreasing radial function satisfying

\[
(21) \quad \int_{\partial B_r(0)} |\nabla \psi| \, d\sigma \leq \int_{B_r(0)} e^\psi \, dx \quad \text{for a.e. } r \in (0, R)
\]

and \( \psi = U_{\lambda_1} = U_{\lambda_2} \) on \( \partial B_R(0) \). Then, it holds either

\[
\int_{B_R(0)} e^\psi \, dx \leq \int_{B_R(0)} e^{U_{\lambda_1}} \, dx \quad \text{or} \quad \int_{B_R(0)} e^\psi \, dx \geq \int_{B_R(0)} e^{U_{\lambda_2}} \, dx.
\]

Moreover, we have

\[
\int_{B_R(0)} (e^{U_{\lambda_1}} + e^{U_{\lambda_2}}) \, dx = 8\pi.
\]

We can now state the Sphere Covering Inequality [16].

Theorem 2.5. Let \( \Omega \subset \mathbb{R}^2 \) be a simply-connected set and let \( w_i \in C^2(\Omega), i = 1, 2 \) be such that

\[
\Delta w_i + e^{w_i} = f_i(x) \quad \text{in } \Omega,
\]

where \( f_2 \geq f_1 \geq 0 \) in \( \Omega \). Suppose

\[
\begin{cases}
  w_2 \geq w_1, \, w_2 \not= w_1 & \text{in } \Omega, \\
  w_2 = w_1 & \text{on } \partial \Omega,
\end{cases}
\]

Then, it holds

\[
\int_{\Omega} (e^{w_1} + e^{w_2}) \, dx \geq 8\pi.
\]

Moreover, if some \( f_i \not= 0 \) then the latter inequality is strict.
The idea is to consider a symmetric rearrangement $\varphi$ of $w_2 - w_1$ with respect to the measures $e^{w_1} \, dx$ and $e^{U_{\lambda_2}} \, dx$ for some suitable $\lambda_2$. Then, by using equation (22) and the properties of the rearrangements, see also Proposition 2.2, it is possible to show that (21) holds true for $\psi = U_{\lambda_1} + \varphi$. Applying then Proposition 2.4 one can deduce that
\[
\int_{\Omega} (e^{w_1} + e^{w_2}) \, dx \geq \int_{B_{\rho}(0)} (e^{U_{\lambda_1}} + e^{U_{\lambda_2}}) \, dx = 8\pi.
\]

See [16] for full details.

**Remark 2.6.** We point out that the latter Sphere Covering Inequality holds as long as the Bol’s inequality of Proposition 2.1 holds. Indeed, if $\Delta w + e^w \geq 0$ in $\Omega$ simply-connected, then the Bol’s and Sphere Covering Inequalities hold on any region $\Omega_1 \subset \Omega$ for general boundary data. In particular, $\Omega_1$ does not need to be simply-connected.

Moreover, for a multiply-connected domain $\Omega$ the Bol’s and Sphere Covering inequalities hold provided we have constant boundary conditions, see [6].

### 3. Asymmetric Sinh-Gordon equation

We are concerned here with the asymmetric Sinh-Gordon equation (1): aim of this section is to present the strategies needed in proofs of both symmetry and uniqueness results of Theorem 1.1, Corollary 1.2 and Theorem 1.4. The first one relies mainly on the Sphere Covering Inequality, see Theorem 2.5. On the other hand, the second one is based on the arguments which yields the Sphere Covering Inequality, which we collected in Section 2.

Let us start with the $\text{supp} \, \mathcal{P} \subset [0, 1]$ case (5) which we recall here for convenience:

\[
\begin{cases}
-\Delta u = \rho \frac{e^u + e^{au}}{\int_{\Omega} (e^u + e^{au}) \, dx} & \text{in } \Omega, \\
u = g(x) \geq 0 & \text{on } \partial \Omega,
\end{cases}
\]

with $a \in (0, 1)$, $\rho > 0$, and $g \in C(\partial \Omega)$. We prove now the uniqueness result.

**Proof of Theorem 1.1.** Let $u_1$ and $u_2$ be solutions of equation (23) satisfying the assumptions of Theorem (1.2). We aim to show that $u_1 \equiv u_2$. We proceed by contradiction by assuming that this is not the case. We start by rewriting equation (23) as

\[
\Delta u + \rho \frac{2e^u}{\int_{\Omega} (e^u + e^{au}) \, dx} = \rho \frac{e^u - e^{au}}{\int_{\Omega} (e^u + e^{au}) \, dx}.
\]

Letting
\[
v = u + \log 2 + \log \rho - \log \left( \int_{\Omega} (e^u + e^{au}) \, dx \right),
\]

with a little abuse of notation we have

\[
\Delta v + e^v = f(u) := \rho \frac{e^u - e^{au}}{\int_{\Omega} (e^u + e^{au}) \, dx}.
\]
It follows from (6) that there exists two regions $\Omega_1, \Omega_2 \subset \Omega$ (not necessarily simply-connected) such that $u_1 > u_2$ in $\Omega_1$, $u_2 > u_1$ in $\Omega_2$, and $u_1 = u_2$ on $\partial \Omega_1 \cup \partial \Omega_2$. We have that $v_1, v_2$ defined by (24) satisfy

$$\Delta v_i + e^{v_i} = f(u_i) \quad \text{in} \quad \Omega.$$  

Moreover

$$v_1 > v_2 \quad \text{in} \quad \Omega_1, \quad v_2 > v_1 \quad \text{in} \quad \Omega_2 \quad \text{and} \quad v_1 = v_2 \quad \text{on} \quad \partial \Omega_1 \cup \partial \Omega_2.$$  

Since $g \geq 0$, both solutions $u_1$ and $u_2$ are positive in $\Omega$ by the maximum principle. By the latter fact it is also easy to see that

$$f(u_1) > f(u_2) > 0 \quad \text{in} \quad \Omega_1$$  

and

$$f(u_2) > f(u_1) > 0 \quad \text{in} \quad \Omega_2.$$  

Therefore, by applying the Sphere Covering Inequality, Theorem 2.5, see also Remark 2.6, we get (observe that $f_i \not\equiv 0$ in our case)

$$\int_{\Omega} (e^{v_1} + e^{v_2}) \, dx \geq \int_{\Omega_1} (e^{v_1} + e^{v_2}) \, dx + \int_{\Omega_2} (e^{v_1} + e^{v_2}) \, dx > 16\pi.$$  

Recalling now the definition of $v$ in (24) and (6) we have

$$4\rho = \frac{2\rho}{\int_{\Omega} (e^{u_1} + e^{au_1}) \, dx} \left( \int_{\Omega} (e^{u_1} + e^{au_1}) \, dx + \int_{\Omega} (e^{v_1} + e^{v_2}) \, dx \right) \geq \frac{2\rho}{\int_{\Omega} (e^{u_1} + e^{au_1}) \, dx} \int_{\Omega} (e^{v_1} + e^{v_2}) \, dx > 16\pi.$$  

Hence $\rho > 4\pi$, which is a contradiction. The proof is now complete. \qed

**Proof of Corollary 1.2.** Without loss of generality that $\Omega$ and $g$ are evenly symmetric with respect to the line $y = 0$. Suppose $u$ is a solution of (5), which is not evenly symmetric about $y = 0$. Then $u_1 = u$ and $u_2(x, y) = u(x, -y)$ are two distinct solutions of (5) satisfying the condition (6). Thus it follows from Theorem 1.1 that $\rho > 4\pi$. \qed

We consider now the general case $\text{supp} \mathcal{P} \subset [-1, 1]$ which yields to (7), i.e.:

\begin{equation}
\begin{cases}
-\Delta u = \rho \frac{e^u - e^{-au}}{\int_{\Omega} (e^u + e^{-au}) \, dx} \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\end{equation}

with $a \in (0, 1), \rho > 0$. We give here the proof of the uniqueness result for the trivial solution $u \equiv 0$. \quad \boxdot

**Proof of Theorem 1.4.** Let $u$ be a solution of (26): we will show that $u \equiv 0$ in $\Omega$. Assume by contradiction this is not the case. The idea is to apply a similar argument as for the Sphere Covering Inequality in Theorem 2.5, see Section 2, to the functions $u$ and $-au$ separately and to derive a contradiction to the assumption on the parameter $\rho$. More precisely, let

\begin{align*}
v_1 &= -au + \log \rho - \log \left( \int_{\Omega} (e^u + e^{-au}) \, dx \right), \\
v_2 &= u + \log \rho - \log \left( \int_{\Omega} (e^u + e^{-au}) \, dx \right).
\end{align*}  

(27)
By using equation (26) we have $v_i$ satisfy
\[ \Delta (v_2 - v_1) + (1 + a) (e^{v_2} - e^{v_1}) = 0 \]
Letting further
\[ w_i = v_i + \log(1 + a), \quad i = 1, 2, \]
we deduce
\[ \Delta (w_2 - w_1) + (e^{w_2} - e^{w_1}) = 0. \]
Moreover, since $u = 0$ on $\partial \Omega$ we get
\[ w_1 = w_2 = \log(1 + a) + \log \rho - \log (\hat{\Omega} (e^u + e^{-au})) \]
on $\partial \Omega$.

It follows there exists at least one region $\tilde{\Omega} \subseteq \Omega$ (not necessarily simply-connected) such that
\[ \begin{cases} w_1 \neq w_2 & \text{in } \tilde{\Omega}, \\ w_1 = w_2 & \text{on } \partial \tilde{\Omega}, \end{cases} \]
and
\[ \Delta (w_2 - w_1) + (e^{w_2} - e^{w_1}) = 0 \quad \text{in } \Omega. \]
We point out $\tilde{\Omega}$ may coincide with $\Omega$. We can further assume $w_2 > w_1$ in $\tilde{\Omega}$, otherwise one can switch the roles of $w_1, w_2$ and repeat all the above argument: it is easy to check all the steps can be carried out in the same way.

Moreover, using equation (26) and the definitions of $w_i$ in (27), (28) we derive that
\[ \Delta v_1 + ae^{v_1} = ae^{v_2} \]
and thus
\[ \Delta w_1 + e^{w_1} = \left( \frac{1}{1 + a} e^{w_1} + ae^{v_2} \right) \geq 0 \quad \text{in } \Omega. \]
We proceed now in the spirit of the Sphere Covering Inequality, see Theorem 2.5, the discussion below it (see also [16]) and Remark 2.6. Let $\lambda_2 > \lambda_1$ be such that $U_{\lambda_2} > U_{\lambda_1}$ in $B_1(0)$ and $U_{\lambda_1} = U_{\lambda_2}$ on $\partial B_1(0)$, where $U_{\lambda}$ is given as in (18), and such that
\[ \int_{\hat{\Omega}} e^{w_1} \, dx = \int_{B_1(0)} e^{U_{\lambda_1}} \, dx. \]
Since $w_1$ satisfies (33) we can take a symmetric equimeasurable rearrangement $\varphi^*$ of $w_2 - w_1$ with respect to the two measures $e^{w_1} \, dx$ and $e^{U_{\lambda_1}} \, dx$, see the discussion after (19). In particular it holds
\[ \int_{\{\varphi^* > t\}} e^{U_{\lambda_1}} \, dx = \int_{\{w_2 - w_1 > t\}} e^{w_1} \, dx. \]
We now first estimate the gradient of the rearrangement by Proposition 2.2, then use equation (32), exploit the properties of the rearrangements and finally use the
equation satisfied by $U_{\lambda_1}$, see below (18), to deduce the following:

$$\int_{\{\varphi^*=t\}} |\nabla \varphi^*| d\sigma \leq \int_{\{w_2-w_1=t\}} |\nabla (w_2 - w_1)| d\sigma$$

$$= \int_{\{w_2-w_1>t\}} (e^{w_2} - e^{w_1}) dx$$

$$= \int_{\{\varphi^*>t\}} e^{U_{\lambda_1}+\varphi^*} dx - \int_{\{\varphi^*=t\}} e^{U_{\lambda_1}} dx$$

$$= \int_{\{\varphi^*>t\}} e^{U_{\lambda_1}+\varphi^*} dx - \int_{\{\varphi^*>t\}} |\nabla U_{\lambda_1}| d\sigma,$$

for a.e. $t > 0$. Therefore, we deduce that

$$\int_{\{\varphi^*=t\}} |\nabla (U_{\lambda_1} + \varphi^*)| d\sigma \leq \int_{\{\varphi^*>t\}} e^{U_{\lambda_1}+\varphi^*} dx,$$

for a.e. $t > 0$. Observe now that $\varphi^*$ is decreasing by construction and thus we derive $U_{\lambda_1} + \varphi^*$ is a strictly decreasing function. Moreover, by the above estimates it holds

$$(34) \quad \int_{\beta B_r(0)} |\nabla (U_{\lambda_1} + \varphi^*)| d\sigma \leq \int_{B_r(0)} e^{U_{\lambda_1}+\varphi^*} dx \quad \text{for a.e. } r > 0.$$

Furthermore, we clearly have

$$\int_{B_1(0)} e^{U_{\lambda_1}+\varphi^*} dx \geq \int_{B_1(0)} e^{U_{\lambda_1}} dx.$$

By the latter estimate and by (34) we can exploit Proposition 2.4 with $\psi = U_{\lambda_1} + \varphi^*$ to get

$$\int_{B_1(0)} e^{U_{\lambda_1}+\varphi^*} dx \geq \int_{B_1(0)} e^{U_{\lambda_2}} dx.$$

Therefore, by construction and by using the latter estimate and by the last property of Proposition 2.4 we finally deduce

$$\int_{\Omega} (e^{w_1} + e^{w_2}) dx = \int_{B_1(0)} (e^{U_{\lambda_1}} + e^{U_{\lambda_1}+\varphi^*}) dx \geq \int_{B_1(0)} (e^{U_{\lambda_1}} + e^{U_{\lambda_2}}) dx = 8\pi.$$

Recall now the definitions of $w_j$ in (27) and (28). We have

$$\frac{\rho(1+a)}{\int_{\Omega} (e^u + e^{-au}) dx} \int_{\Omega} (e^u + e^{-au}) dx \geq 8\pi.$$

It follows that

$$\frac{8\pi}{1+a} \leq \frac{\rho}{\int_{\Omega} (e^u + e^{-au}) dx} \int_{\Omega} (e^u + e^{-au}) dx \leq \rho.$$

Moreover, going back into the above argument it is possible to show that the latter inequality is strict. Indeed, the equality would yield the equality in (34) which corresponds to equality in the Bol’s inequality in Proposition 2.1 for $w_2$. Then $w_2$ would satisfy $\Delta w_2 + e^{w_2} = 0$: one can check this is not the case.

This contradicts the fact that $\rho \leq \frac{8\pi}{1+a}$ by assumption. We conclude that necessarily $u \equiv 0$ in $\Omega$ as desired. $\square$
4. Cosmic string equation

In this section we are concerned with the cosmic string equation (8), which we recall below:

\[
\begin{cases}
-\Delta u = e^{au} + h(x) e^u & \text{in } \Omega, \\
u = g(x) \geq 0 & \text{on } \partial \Omega,
\end{cases}
\]

with \( a > 0 \) and \( h \) as in (9). We show here how we can suitably modify the latter equation in order to exploit the Sphere Covering Inequality, Theorem 2.5, and prove the no intersection property of Theorem 1.8.

**Proof of Theorem 1.8.** We start by considering the case \( a > 1 \). Let \( u_1 \) and \( u_2 \) be two solutions of (35) with \( a > 1, N \geq 0 \) satisfying (12). We proceed by contradiction. Suppose there exists \( \Omega_1, \Omega_2 \subset \Omega \) (not necessarily simply-connected) such that

\[
u_1 > u_2 \quad \text{in } \Omega_1 \quad \text{and} \quad u_2 > u_1 \quad \text{in } \Omega_2.
\]

We write (35) in the form

\[
\Delta u + 2e^{au} = e^{au} - h(x) e^u.
\]

By multiplying both sides of the latter equation by \( a \) and by letting

\[
v = au + \log (2a),
\]

with a little abuse of notation \( v \) satisfies

\[
\Delta v + e^v = f(u) := a(e^{au} - h(x) e^u).
\]

Let \( v_1, v_2 \) defined by (37) \((u \text{ replaced by } u_1 \text{ and } u_2, \text{ respectively})\). Then we have

\[
\Delta v_i + e^{v_i} = f(u_i) \quad \text{in } \Omega.
\]

Furthermore, we get

\[
v_1 > v_2 \quad \text{in } \Omega_1, \quad v_2 > v_1 \quad \text{in } \Omega_2 \quad \text{and} \quad v_1 = v_2 \quad \text{on } \partial \Omega_1 \cup \partial \Omega_2.
\]

Since \( g \geq 0 \), both solutions \( u_1 \) and \( u_2 \) are positive inside \( \Omega \) by the maximum principle. Moreover, observe that \( h(x) \leq 1 \) still by applying the maximum principle to the Green’s function and by the definition of \( h \), see (9). By exploiting the latter properties it is easy to see that

\[
f(u_1) > f(u_2) > 0 \quad \text{in } \Omega_1 \quad \text{and} \quad f(u_2) > f(u_1) > 0 \quad \text{in } \Omega_2.
\]

By the Sphere Covering Inequality (Theorem 2.5), see also Remark 2.6, we conclude that (observe that \( f_i \neq 0 \))

\[
\int_\Omega (e^{v_1} + e^{v_2}) \, dx \geq \int_{\Omega_1} (e^{v_1} + e^{v_2}) \, dx + \int_{\Omega_2} (e^{v_1} + e^{v_2}) \, dx > 16 \pi.
\]

Using the expression of \( v \) in (36) we deduce

\[
2a \int_\Omega (e^{au_1} + e^{au_2}) \, dx > 16 \pi,
\]

which contradicts the assumption

\[
\int_\Omega (e^{au_1} + e^{au_2}) \, dx \leq \frac{8 \pi}{a}.
\]
Therefore, we get the desired property. For what concerns the case $a < 1$ we write (35) in the form
\[ \Delta u + 2e^u = (e^u - e^{au}) + (e^u - h(x)e^u).\]
The argument is then developed as before so we skip the details. The proof is now complete. $\square$

Proof of Corollary 1.9. Without loss of generality that $\Omega$ and $g$ are evenly symmetric with respect to the line $y = 0$. Observe that the associated Green’s function (and hence $h$, see (9)) is evenly symmetric with respect to the line $y = 0$. We consider just the case $a > 1$ since for $a < 1$ one can proceed in the same way. Suppose $u$ is a solution of (5) satisfying (13), which is not evenly symmetric about $y = 0$.

Then $u_1 = u$ and $u_2(x,y) = u(x,-y)$ are two distinct intersecting solutions of (8). It follows from Theorem 1.8 that
\[ 2\int_\Omega e^{au} \, dx = \int_\Omega (e^{au_1} + e^{au_2}) \, dx > \frac{8\pi}{a}. \]
which is a contradiction to (13). $\square$

5. Toda system

In this section we consider the class of Liouville-type systems as in (14), i.e.:
\[
\begin{cases}
-\Delta u_1 = Ae^{u_1} - Be^{u_2} & \text{in } \Omega, \\
-\Delta u_2 = B'e^{u_2} - A'e^{u_1} & \\
u_1 = u_2 = g(x) & \text{on } \partial\Omega,
\end{cases}
\]
where $A, A', B, B'$ satisfy the conditions (15). We introduce here the argument which yields to the uniqueness result stated in Theorem 1.11. The latter is based Sphere Covering Inequality, which we collected in Section 2.

Proof of Theorem 1.11. Let $(u_1, u_2)$ be a solution of (38). We will prove that there exists a unique $u$ solving a mean field equation as stated in Theorem 1.11 such that $u_1 \equiv u_2 \equiv u$ in $\Omega$. Assume by contradiction $u_1 \not\equiv u_2$. As in the proof of Theorem 1.4, the strategy is to follow the argument of the Sphere Covering Inequality in Theorem 2.5, see Section 2, applied to the functions $u_1$ and $u_2$. We start by recalling that the coefficients in (38) are such that $A + A' = B + B' := M$.

Therefore, by using system (38) we have
\[ \Delta(u_2 - u_1) + M(e^{u_2} - e^{u_1}) = 0. \]
Letting
\[ w_i = u_i + \log M, \quad i = 1, 2, \]
we deduce that
\[ \Delta(w_2 - w_1) + (e^{w_2} - e^{w_1}) = 0. \]
Moreover, due to the boundary conditions we get
\[ w_1 = w_2 = \log M + g(x) \quad \text{on } \partial\Omega. \]
It follows that there exists at least one region \( \tilde{\Omega} \subseteq \Omega \) (not necessarily simply-connected) such that

\[
\begin{cases}
    w_1 \neq w_2 & \text{in } \tilde{\Omega}, \\
    w_1 = w_2 & \text{on } \partial\tilde{\Omega},
\end{cases}
\]

and

\[
\Delta (w_2 - w_1) + (e^{w_2} - e^{w_1}) = 0 \quad \text{in } \Omega.
\]

We can further assume \( w_2 > w_1 \) in \( \tilde{\Omega} \), otherwise one can switch the roles of \( w_1, w_2 \) and repeat all the above steps.

Furthermore, using the first equation in (38), the definitions of \( w_i \) in (28) and the fact that \( M = A + A' \) we get

\[
\Delta u_1 + Ae^{u_1} = Be^{u_2}
\]

and hence

\[
\Delta w_1 + e^{w_1} = \left( \frac{A'}{A + A'} e^{w_1} + Be^{w_2} \right) \geq 0 \quad \text{in } \Omega.
\]

In the above two steps we assumed \( A > 0 \); however, the latter property holds true even if \( A = 0 \) by simple manipulations. One can now follow the strategy of the Sphere Covering Inequality, see Theorem 2.5, the discussion below it (see also [16]) and Remark 2.6. The argument is very similar to the proof of Theorem 1.4 so we will skip the details, referring to the latter proof for full details. Let \( \lambda_2 > \lambda_1 \) be such that \( U_{\lambda_2} > U_{\lambda_1} \) in \( B_1(0) \) and \( U_{\lambda_1} = U_{\lambda_2} \) on \( \partial B_1(0) \), where \( U_\lambda \) is given as in (18), and such that

\[
\int_{\tilde{\Omega}} e^{w_1} \, dx = \int_{B_1(0)} e^{U_{\lambda_1}} \, dx.
\]

Recalling (44) we can take a symmetric equimeasurable rearrangement \( \varphi^* \) of \( w_2 - w_1 \) with respect to the two measures \( e^{w_1} \, dx \) and \( e^{U_{\lambda_1}} \, dx \), see the discussion after (19). Reasoning as in the proof of Theorem 1.4 we estimate the gradient of the rearrangements by Proposition 2.2, then use equation (43), exploit the properties of the rearrangements and use the equation satisfied by \( U_{\lambda_1} \), see below (18), to get

\[
\int_{\partial B_r(0)} |\nabla (U_{\lambda_1} + \varphi^*)| \, d\sigma \leq \int_{B_1(0)} e^{U_{\lambda_1} + \varphi^*} \, dx \quad \text{for a.e. } r > 0.
\]

Furthermore, \( U_{\lambda_1} + \varphi^* \) is a strictly decreasing function. Hence, applying Proposition 2.4 to \( \psi = U_{\lambda_1} + \varphi^* \) we deduce

\[
\int_{B_1(0)} e^{U_{\lambda_1} + \varphi^*} \, dx \geq \int_{B_1(0)} e^{U_{\lambda_2}} \, dx.
\]

Therefore, by construction and by using the latter estimate and by the last property of Proposition 2.4 we get

\[
\int_{\tilde{\Omega}} (e^{w_1} + e^{w_2}) \, dx = \int_{B_1(0)} \left( e^{U_{\lambda_1}} + e^{U_{\lambda_1} + \varphi^*} \right) \, dx \geq \int_{B_1(0)} \left( e^{U_{\lambda_1}} + e^{U_{\lambda_2}} \right) \, dx = 8\pi.
\]

Going back to the definitions of \( w_i \) in (39) we have

\[
M \int_{\tilde{\Omega}} (e^{u_1} + e^{u_2}) \, dx \geq 8\pi.
\]
It follows that
\[ \frac{8\pi}{M} \leq \int_{\Omega} (e^{u_1} + e^{u_2}) \, dx \leq \int_{\Omega} (e^{u_1} + e^{u_2}) \, dx. \]
Arguing as in the proof of Theorem 1.4 it is possible to show that the latter inequality is strict. Since by assumption
\[ (45) \int_{\Omega} (e^{u_1} + e^{u_2}) \, dx \leq \frac{8\pi}{M}, \]
a contradiction arises. We conclude \( u_1 \equiv u_2 \) in \( \Omega \). Letting \( u := u_1 = u_2 \) we use system (38) to get
\[ \begin{cases} -\Delta u = D e^u & \text{in } \Omega, \\ u = g(x) & \text{on } \partial \Omega, \end{cases} \]
where we recall \( D := A - B = A' - B' \). Observe that by exploiting (45) and \( M := A + A' = B + B' \) it holds
\[ \int_{\Omega} D e^u \, dx \leq 4\pi \frac{D}{M} = 4\pi \frac{A - B}{A + A'} < 4\pi. \]
Since \( \Omega \) is simply-connected and the latter bound holds true, by the Sphere Covering Inequality of Theorem 2.5 we deduce that \( u \) is unique. This concludes the proof of Theorem 1.11. \( \square \)

We conclude this section by giving the sketch of the proof of the uniqueness result concerning the following singular Liouville-type system version, see Theorem 1.13:
\[ (46) \begin{cases} -\Delta u_1 = A e^{u_1} - B e^{u_2} - 4\pi \alpha \delta_0 & \text{in } \Omega, \\ -\Delta u_2 = B' e^{u_2} - A' e^{u_1} - 4\pi \alpha \delta_0 & \text{in } \Omega, \\ u_1 = u_2 = g(x) & \text{on } \partial \Omega, \end{cases} \]

**Proof of Theorem 1.13.** Let \((u_1, u_2)\) be a solution of (46) with \( \alpha \geq 0 \). By using the Green’s function \( G_0 \) with pole in 0 as in (10) we desingularize the problem by setting
\[ \tilde{u}_i(x) = u(x) + 4\pi \alpha G_0(x). \]
Indeed, (46) is equivalent to
\[ (47) \begin{cases} -\Delta \tilde{u}_1 = A h(x) e^{\tilde{u}_1} - B h(x) e^{\tilde{u}_2} & \text{in } \Omega, \\ -\Delta \tilde{u}_2 = B' h(x) e^{\tilde{u}_2} - A' h(x) e^{\tilde{u}_1} & \text{in } \Omega, \\ \tilde{u}_1 = \tilde{u}_2 = g(x) & \text{on } \partial \Omega, \end{cases} \]
where
\[ (48) h(x) = e^{-4\pi \alpha G_0(x)}. \]
Observe that
\[ h > 0 \quad \text{in } \Omega \setminus \{0\} \quad \text{and} \quad h(x) \sim |x|^{2\alpha} \quad \text{near } 0. \]
Assume now by contradiction that \( \tilde{u}_1 \not\equiv \tilde{u}_2 \) and suppose without loss of generality that \( \tilde{u}_2 > \tilde{u}_1 \) in \( \tilde{\Omega} \subseteq \Omega \). Recall that \( A + A' = B + B' := M \). Therefore, by (47) we have
\[ \Delta (\tilde{u}_2 - \tilde{u}_1) + M h(x) (e^{\tilde{u}_2} - e^{\tilde{u}_1}) = 0. \]
Observe that by the maximum principle applied to the Green’s function and by the definition of $h$, see (48), we have $h(x) \leq 1$. Since $\tilde{u}_2 > \tilde{u}_1$ in $\tilde{\Omega}$ we deduce
\[
\Delta (\tilde{u}_2 - \tilde{u}_1) + M (e^{\tilde{u}_2} - e^{\tilde{u}_1}) \geq 0 \quad \text{in } \tilde{\Omega}.
\]
One can then follow the same strategy of the proof of Theorem 1.11 to get a contradiction. It follows that $\tilde{u}_1 \equiv \tilde{u}_2 := \tilde{u}$ and $\tilde{u}$ satisfies
\[
\begin{cases}
-\Delta \tilde{u} = Dh(x)e^{\tilde{u}} & \text{in } \Omega, \\
\tilde{u} = g(x) & \text{on } \partial \Omega,
\end{cases}
\]
where $D := A - B = A' - B'$. Arguing as in the proof of Theorem 1.11 we deduce that $\tilde{u}$ is unique and the proof is concluded. □

Acknowledgements

The authors would like to thank Prof. G. Tarantello and Dr. W. Yang for the discussions concerning the topic of this paper.

References


**Changfeng Gui, Department of Mathematics, University of Texas at San Antonio, Texas, USA**

*E-mail address:* changfeng.gui@utsa.edu

**Aleks Jevnikar, University of Rome ‘Tor Vergata’, Via della Ricerca Scientifica 1, 00133 Roma, Italy**

*E-mail address:* jevnikar@mat.uniroma2.it

**Amir Moradifam, Department of Mathematics, University of California, Riverside, California, USA**

*E-mail address:* moradifam@math.ucr.edu